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13. ABSTRACT (Maximum 200 words)

The eight publications produced by the project established a number of basic results in the theory of viscosity solutions of fully nonlinear partial differential equations of first and second order in finite and infinite dimensions. These equations arise in the dynamic programming theory of control and differential games (the finite dimensional theory for ode and the infinite dimensional theory for pde dynamics). Being fully nonlinear, the equations do not typically admit regular or classical solutions, and the appropriate notion is that of "viscosity solutions". Two major advances in the first order infinite dimensional case consisted of determining the precise notion appropriate to a class of infinite dimensional problems with "unbounded terms" arising from the pde dynamics, and the examination of a limit case in which the value function is not a solution, but the maximal subsolution. Significant contributions to the second order theory include a new exposition of the finite dimensional theory based on results from previous funding, an infinite dimensional generalization of the foundational result used in this expostion, and the extension of the theory to second order equations in infinite dimensions with unbounded first order terms.

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VISCOSITY SOLUTIONS OF FULLY NONLINEAR EQUATIONS

FINAL REPORT

MICHAEL G. CRANDALL

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U. S. ARMY RESEARCH OFFICE

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UNIVERSITY OF CALIFORNIA, SANTA BARBARA

A. The Problem Studied: Viscosity Solutions of Fully Nonlinear Equations

Let us set the general mathematical arena in which this project was conducted. Questions concerning fully nonlinear partial differential equations of first and second order in finite and infinite dimensions were resolved. These equations are significant in the theory of control and differential games in finite and infinite dimensions, as well as many other areas. Progress was made in various technical settings corresponding to a broad range of different results.

We begin by writing a "general" case. Let H be a real separable Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and

$$F: H \times \mathbb{R} \times H \times \mathcal{S}(H) \to \mathbb{R}$$

where $\mathcal{S}(H)$ denotes the space of bounded self-adjoint operators on H. We say that F is *proper* if it both satisfies the monotonicity requirement

$$F(x,r,p,X) \le F(x,s,p,X) \quad \text{for} \quad r \le s,\ x,p \in H \text{ and } X \in \mathcal{S}(H)$$

and is *degenerate elliptic*; i.e.

$$F(x,r,p,X) \le F(x,r,p,Y) \quad \text{for} \quad x,p \in H, r \in \mathbb{R}, X,Y \in \mathcal{S}(H) \text{ and } Y \le X$$

where the usual ordering is used on $\mathcal{S}(H)$. In the most important special case $H = \mathbb{R}^n$, we write $\mathcal{S}(n)$ in place of $\mathcal{S}(\mathbb{R}^n)$ to denote the set of real symmetric $n \times n$ matrices.

Associated with a proper mapping F is the *fully nonlinear* second order partial differential equation

$$F(x,u,Du,D^2u) = 0 \tag{0.1}$$

and the associated evolution problem

$$u_t + F(x,u,Du,D^2u) = 0. \tag{0.2}$$

Here Du stands for the gradient and D^2u for the second derivative (the matrix of second order partials when $H = \mathbb{R}^n$) of the real-valued function u. When F is independent of p the equation is of first order ($F(x,r,p,X) = F(x,r,p)$ is always proper if it is nondecreasing in r). For some purposes we may regard (0.2) as a special case of (0.1) by merely regarding the pair (t,x) as a "new" x. In this spirit, the following definitions apply to both (0.1) and (0.2). If $\mathcal{O}$ is a locally closed subset of H and $\hat{x} \in \mathcal{O}$, we define

(0.3)

$$J^{2,+}_{\mathcal{O}}u(x) = \left\{ (p,X) : u(x) \le u(\hat{x}) + \langle p, x-\hat{x}\rangle + \frac{1}{2}\langle X(x-\hat{x}), x-\hat{x}\rangle + o(\|x-\hat{x}\|^2) \right\},$$

where the "$o(\|x-\hat{x}\|^2)$" is to hold as $\mathcal{O} \ni x \to \hat{x}$. We put $J^{2,-}_{\mathcal{O}}u(\hat{x}) = -J^{2,+}_{\mathcal{O}}(-u)(\hat{x})$. If $\hat{x}$ is interior to $\mathcal{O}$, these sets do not depend on $\mathcal{O}$ and are written $J^{2,+}u(\hat{x}), J^{2,-}u(\hat{x})$. Alternatively, one has

$$(0.4)\qquad J_{\mathcal{O}}^{2,+}u(\hat{x}) = \left\{(D\varphi(\hat{x}), D^2\varphi(\hat{x})) : \varphi \text{ is } C^2 \text{ near } \hat{x} \text{ and } u-\varphi \text{ has a maximum at } \hat{x}\right\},$$

etc.

A *viscosity* subsolution (supersolution) of $F = 0$ on $\mathcal{O}$ is an upper-semicontinuous (respectively, lower-semicontinuous) function u on $\mathcal{O}$ such that

$$F(x, u(x), p, X) \le 0 \quad \text{for} \quad x \in \mathcal{O} \text{ and } (p, X) \in J_{\mathcal{O}}^{2,+}u(x)$$

(respectively,

$$F(x, u(x), p, X) \ge 0 \quad \text{for} \quad x \in \mathcal{O} \text{ and } (p, X) \in J_{\mathcal{O}}^{2,-}u(x)).$$

A viscosity solution is a continuous function which is simultaneously a subsolution and a supersolution.

In the infinite dimensional case, one is also interested in choices of F with "unbounded" behavior in x, p, or X, which means that F is not defined in full neighborhoods and is not locally bounded in the variable indicated. For example, if A is a linear, unbounded, maximal monotone operator in H (typically the generator of a semigroup assoicated with a partial differential equation of evolution), then

$$F(x, r, p) = r + \langle Ax, p\rangle + G(x, p)$$

is a first order operator with the unbounded term $\langle Ax, p\rangle$ (or, perhaps, $\langle x, A^*p\rangle$), which is unbounded in x or p depending on the way one chooses to wrtie the expression. In unbounded cases, the notion of solution discussed above must be modified in appropriate ways, and a serious and subtle part of the work is to discover the appropriate notions.

The project focused on developing the theory of the various types of equations referred to able. The main issues were: appropriate notions of solutions, existence of solutions, uniqueness of solutions and other properties of solutions.

B. Summary of the Most Important Results

The bibliographic reference [Pn] refers to publication number n in the list (immediately following) of publications generated by the project. Other references are not given, as at least the introductions of the publications quoted should be consulted for a more complete explanation of the work they contain and bibiliographic questions.

We begin by summarizing the contributions of the project to the finite dimensional theory of viscosity solutions.

[P1] contains a clear and simplified organization of the theory of viscosity solutions of fully nonlinear partial differential equations of second order together with some new results. This presentation was made possible by previous foundational theoretical work supported by ARO funding. The paper has made a significant impact in many areas by making available an accessible overview of the subject. It has been used by engineers, experts in mathematical finance and others as well as mathematicians.

The work [P3], begun under previous funding, examines the effect of degeneracies on the uniqueness theory of viscosity solutions of second order equations. It establishes that one must procede with care here, and obtains a necessary and sufficient condition for uniqueness for a "linear" equation in one space dimension when the coefficient of the second order term vanishes at a single point and goes on to consider more complex situations. It shows that naive definitions of viscosity solutions can suffer nonuniqueness even in the presence of a unique strong solution.

We turn to a summary of the contributions to the infinite dimensional theory.

[P4] and [P5] concern first order Hamilton-Jacobi equations in infinite dimensional spaces. These equations are the type which arise when dynamic programming is applied to control and differential game problems for pde's. Value functions for control and differential games problems satisfy equations of this sort in the viscosity sense. In [P4] methods introduced by Tataru which extend and simplify the subject in under certain technical conditions are reorganized and made transparent. It is always an issue in the infinite dimensional case exactly how to define viscosity solutions of the equations. The equations involve semigroup generators from the pde's driving the dynamics of the problem and hence terms which are not everywhere defined. In [P4] Tataru's more complex interpretation is replaced by relaxed Dini type derivatives along the trajectories of the semigroups. A complex interpolation argument is replaced by a summarizing "doubling lemma" and with this in hand, the theory can proceed much as in the finite dimensional case. The resulting theory is elegant and relatively transparent. The dynamics can involve even nonlinear semigroup generators, one of Tataru's significant contributions.

In the paper [P5] the theoretical issues addressed are illustrated by the following concrete situation. Suppose dynamics for a control problem are given by the heat equation with the control appearing as a forcing term. The cost is given by a discounted cost functional of the state (the temperature distribution) integrated over a trajectory with respect to time (we take the infinite horizon problem). Theory developed under previous funding and in [P4] above would allow the control to take values in the Lebesgue space L^2 and the cost functional to use measurements of the state in L^2. However, it is most natural to use controls in H^{-1} and to allow the cost to measure the gradient of the state. In [P5] it is shown that this is an extreme situation and in general the value function does not appear to be a solution of the Hamilton-Jacobi-Bellman equation. However, it can be identified as the maximal subsolution of the Hamilton-Jacobi-Bellman equation. The example illustrating the difficulties is straightforward if subtle, while the other arguments are lengthy, technical and ad hoc. If the cost is convex or weakly continuous, the difficulties are ameliorated, as is made clear.

The paper [P6] returns to foundational questions in infinite dimensions, this time addressed to the infinite dimensional case (corresponding to stochastic control of pde's). The proof by P. L. Lions of a fundamental lemma had a gap which is illuminated by an example in [P6]. Then a complete and much simplified proof of the result is given and generalized in several ways.

The paper [P8] contains the major results of the thesis of A. Swiech, a student supported on the grant. Illuminated by the results of [P6], Swiech was able to develop a theory of second order equations in infinite dimensions, corresponding to stochastic control of problems with whose dynamics involve partial differential equations. Swiech treated a fairly general case with unbounded first order terms, as well as some cases with unbounded second order terms. Existence and uniqueness was established in a setting more congenial and familiar than competing theories.

[P7], by Swiech and Kocan (a second student supported by the grant who should obtain his Ph.D. this year) obtains an important result in perturbed optimization. Perturbed optimization techniques are used in essential and delicate ways in the infinite dimensional theory. This result will be used by Kocan in his thesis, in which he will develop a theory covering other important second order equations.

Finally we mention [P1], which sits by itself. It is a study of interpolation questions arising, roughly speaking, from the interaction of order preservation and either L^∞ or L^1 contraction. Moreover, properties of generators of evolutions are related to properties of this sort of the semiflows they generate.

C. List of Publications and Technical Reports

[P1] Crandall, M. G., and Ph. Benilan, Completely Accretive Operators, *semigroup theory and evolution equations*, P. Clement, E. Mitidieri and B. de Pagter, eds., Lecture Notes in Mathematics 135, Marcel Dekker, New York, 1991, pp. 41-76.

[P2] Crandall, M. G., H. Ishii and P. L. Lions, User's Guide to viscosity solutions of second order partial differential equations, *Bulletin of the American Mathematical Society* 27 (1992), 1-67.

[P3] Crandall, M. G., and Z. Huan, Viscosity solutions of a degenerate linear equation, *Differential and Integral Equations* 5 (1992), 1247-1265.

[P4] Crandall, M. G., and P. L. Lions, Hamilton-Jacobi equations in infinite dimensions: Part VI. Nonlinear A and Tataru's method refined, *evolution equations, control theory and biomathematics*, Ph. Clement and G. Lumer, eds., Lecture Notes in Pure and Applied Mathematics 116, Marcel Dekker, New York, 1993, 51-89.

[P5] Crandall, M. G., and P. L. Lions, Hamilton-Jacobi equations in infinite dimensions: Part VII. The HJB equation is not always satisfied, *Journal of Functional Analysis*, to appear.

[P6] Crandall, M. G., M. Kocan and A. Swiech, On partial sup-convolutions, a lemma of P. L. Lions and viscosity solutions in Hilbert spaces, to appear.

[P7] Perturbed optimization on product spaces, *Nonlinear Analysis Theory, Methods and Applications*, to appear.

[P8] Swiech, A., Solutions of fully nonlinear second order equations in Hilbert spaces with unbounded terms, preprint.

D. List of All Participating Scientific Personnel

M. Kocan, Graduate student, University of California, Santa Barbara (Ph. D. expected 8/94)

P. L. Lions, University of Paris, Dauphine

A. Swiech, Graduate student, University of California, Santa Barbara (Ph. D. 8/93 – currently Assistant Professor, Georgia Institute of Technology)